Anti-self-dual Riemannian metrics without Killing vectors, can they be realized on K3?

A A Malykh[1], **Y Nutku**[2] **and M B Sheftel**[1,2]

[1] Department of Higher Mathematics, North Western State Technical University, Millionnaya St. 5, 191186, St. Petersburg, Russia
[2] Feza Gürsey Institute, PO Box 6, Cengelkoy, 81220 Istanbul, Turkey
E-mail: specarm@online.ru, nutku@gursey.gov.tr, sheftel@gursey.gov.tr

Abstract
Explicit Riemannian metrics with Euclidean signature and anti-self dual curvature that do not admit any Killing vectors are presented. The metric and the Riemann curvature scalars are homogenous functions of degree zero in a single real potential and its derivatives. The solution for the potential is a sum of exponential functions which suggests that for the choice of a suitable domain of coordinates and parameters it can be the metric on a compact manifold. Then, by the theorem of Hitchin, it could be a class of metrics on $K3$, or on surfaces whose universal covering is $K3$.

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We present Riemannian metrics with anti-self-dual curvature that admit no Killing vectors. Our motivation to study this problem has been $K3$ which is the most important gravitational instanton [1]-[3]. It is necessary that the metric on $K3$ should not admit any continuous symmetries. Our solutions do satisfy this criterion which is necessary but not sufficient since $K3$ is a *compact* 4-dimensional Riemannian manifold. In this note all our considerations will be local and we shall not discuss the global problem of compactness. The metric

$$ds^2 = u_{i\bar{k}}\, d\zeta^i d\bar{\zeta}^k \tag{1}$$

must be hyper-Kähler. It has been over a century since Kummer [5] introduced $K3$ as a quartic surface in CP^3 and half a century since Calabi [6] pointed out that the Kähler potential satisfies the elliptic complex Monge-

Ampère equation

$$u_{1\bar{1}}u_{2\bar{2}} - u_{1\bar{2}}u_{\bar{1}2} = 1 \tag{2}$$

hereafter to be referred to as CMA_2. Then the metric has vanishing first Chern class [4] and therefore satisfies the Euclidean Einstein field equations.

Yau [7] has given an existence and uniqueness proof but so far there are no explicit exact solutions of CMA_2 that do not admit any continuous symmetries. Another testament to the difficulties encountered in dealing with complex Monge-Ampère equations lies in the fact that its homogeneous version replaces Laplace's equation as the fundamental equation governing functions of many complex variables [8]. The principal difficulty in constructing solutions of CMA_2 that would describe $K3$ lies in the requirement that the Kähler metric (1) must not admit any Killing vectors. In the language of differential equations such solutions are known as non-invariant solutions of (2). Recently we suggested that the method of group foliation [9, 10, 11] can serve as a regular tool for finding non-invariant solutions of non-linear partial differential equations. Group foliation was carried out for CMA_2 and the Boyer-Finley equations in [12] and [13] respectively using their infinite symmetry groups [14].

However, in this note we shall adopt a different approach which turned out to be fruitful specifically for CMA_2, to find an explicit metric without any Killing vectors that has anti-self-dual curvature. We emphasize at the outset that the class of solutions we are considering here is not the full set of solutions of CMA_2. In our approach we start with the Mason-Newman [15] Lax pair and supplement the Lax equations with two more linear equations such that CMA_2 emerges as an algebraic compatibility condition. The would be Baker-Akhiezer function in the standard Lax approach is now regarded as a complex potential. Choosing symmetry characteristics [16] of CMA_2 for the real and imaginary parts of this potential we arrive at an over-determined set of *linear* equations satisfied by one real potential. This system is the image of CMA_2 supplemented by some differential constraints after performing a Legendre transformation. Its solution gives exact solutions of the Euclidean Einstein equations with anti-self-dual Riemann curvature 2-form. Here we shall present only the final results and postpone the detailed derivation to a future publication, a preliminary account of which can be found in [17].

We use the Euclidean Newman-Penrose formalism [18], [19] to write the metric in the form

$$ds^2 = l \otimes \bar{l} + \bar{l} \otimes l + m \otimes \bar{m} + \bar{m} \otimes m \tag{3}$$

where the co-frame $\omega^a = \{l, \bar{l}, m, \bar{m}\}$ is given by

$$
\begin{aligned}
l &= \frac{1}{v\left[C(C^2-|A|^2)\right]^{1/2}}\left[C(Cdz^1+Bdz^2)+\bar{A}(Cd\bar{z}^1+\bar{B}d\bar{z}^2)\right] \qquad (4)\\
m &= \frac{(C^2-|A|^2)^{1/2}}{v\,C^{1/2}}dz^2 \qquad (5)
\end{aligned}
$$

and A, B, C, v are *a priori* functions of all coordinates. The first three are expressed in terms of v

$$
\begin{aligned}
A &= v^2 + v_1^2 - ivv_2,\\
B &= v_2 v_{\bar{1}} - iv(v_1 - v_{\bar{1}}) \qquad (6)\\
C &= v^2 + |v_1|^2
\end{aligned}
$$

where v is a real-valued potential. Then the anti-self-duality equations and therefore the Euclidean Einstein field equations reduce to a system of over-determined *linear* equations

$$
\begin{aligned}
v_{1\bar{1}} + v &= 0\\
v_{11} + v - iv_2 &= 0\\
v_{1\bar{2}} + i(v_{\bar{1}} - v_1) &= 0 \qquad (7)\\
v_{2\bar{2}} + i(v_{\bar{2}} - v_2) &= 0
\end{aligned}
$$

which together with their complex conjugates make up 6 real equations. It is interesting to note that by virtue of (7) we have $A_{\bar{1}} = -iB,\ C_1 = iB$ in terms of the original potentials. We further note that there are no first order equations implied by (7). Since the conditions for invariant solutions of a differential equation are first order equations, the general solution of the system (7) must correspond to non-invariant solutions of CMA_2. This implies that there are no Killing vectors in the metric.

It can be verified directly that the system (6), (7) gives an anti-self-dual Riemann curvature 2-form

$$
\Omega^a_{\ b} = -{}^*\Omega^a_{\ b}, \qquad \Omega^a_{\ b} = \frac{1}{2}R^a_{\ bcd}\,\omega^c\wedge\omega^d \qquad (8)
$$

where $*$ is the Hodge star operator. Ricci-flatness follows by virtue of the first Bianchi identity.

We shall consider here a particular solution of (7) which can be given in a finite form, so that it does not contain infinite series or an integral. This is

$$v = \sum_{j=-\infty}^{\infty} \exp\Big\{2\,\mathrm{Im}\Big(\left[\alpha_j^2(s_j^2+1)+1\right] z^2\Big)\Big\} \Big\{ \tag{9}$$

$$\exp\Big[\,2s_j\mathrm{Re}(\alpha_j z^1)\Big]\mathrm{Re}\Big\{D_j \exp\Big[2i\,\Big[\mathrm{Im}\big(\alpha_j z^1\big) - 2s_j\mathrm{Re}(\alpha_j^2 z^2)\Big]\Big]\Big\}$$

$$+\exp\Big[-2s_j\mathrm{Re}(\alpha_j z^1)\Big]\mathrm{Re}\Big\{E_j \exp\Big[2i\,\Big[\mathrm{Im}\big(\alpha_j z^1\big) + 2s_j\mathrm{Re}(\alpha_j^2 z^2)\Big]\Big]\Big\}\,\Big\}$$

where α_j, D_j, E_j are arbitrary complex constants and $s_j = \sqrt{1-1/|\alpha_j|^2}$. The general solution of the linear system (7) can be given by the corresponding infinite series, or by an integral representation for the case of the continuous spectrum with α_j changed to α, s_j to $s=\sqrt{1-1/|\alpha|^2}$, D_j, E_j to $D(\alpha,\bar{\alpha}), E(\alpha,\bar{\alpha})$ respectively and the sum changed to a double integral with respect to $\alpha, \bar{\alpha}$.

The locus of possible singularities of the curvature scalars is given by the first order partial differential equation

$$v\,\Big[\,(v_1 - v_{\bar{1}})^2 + |v_2|^2\,\Big] - iv^2(v_2 - v_{\bar{2}}) - i(v_2 v_{\bar{1}}^2 - v_{\bar{2}} v_1^2) = 0 \tag{10}$$

which are also singularities of the metric. Imposing equation (10) in addition to the system of equations (7) results in further relations between the metric coefficients

$$A = \lambda C, \qquad B = \mu C$$

where λ and μ are arbitrary complex constants. Further analysis of the compatibility conditions arising from (7) and (10) gives the general form of the singular solution

$$v = \mathrm{Re}\Big\{D\exp\Big\{\,\alpha(s+1)z^1 + \bar{\alpha}(s-1)\bar{z}^1 \tag{11}$$

$$-i\big[\alpha^2(s+1)^2+1\big]\,z^2 + i\big[\bar{\alpha}^2(s-1)^2+1\big]\,\bar{z}^2\Big\}\Big\}$$

where

$$\lambda = -\frac{\alpha}{\bar{\alpha}}\,, \quad \mu = 2i\alpha s, \quad s = \sqrt{1-\frac{1}{|\alpha|^2}}$$

and D is an arbitrary complex constant. We conclude that curvature singularities *related to the choice of* v will arise if and only if it is given by (11).

Any solution for v of the form (9) when substituted into (6) gives us an explicit form of the metric

$$\begin{aligned} ds^2 &= \frac{1}{v^2(C^2-|A|^2)}\Big[A(Cdz^1+Bdz^2)^2+\bar{A}(Cd\bar{z}^1+\bar{B}d\bar{z}^2)^2 \\ &\quad +\frac{1}{C}(C^2+|A|^2)|Cdz^1+Bdz^2|^2\Big]+\frac{1}{v^2C}(C^2-|A|^2)|dz^2|^2 \end{aligned} \tag{12}$$

which is an exact solution of the anti-self-duality equations (8) and therefore the Einstein field equations with Euclidean signature. The fact that it admits no Killing vectors follows from the non-trivial dependence of v on all four coordinates provided we keep a minimum of two terms in the sum (9).

We shall not discuss whether, or not our solution describes a compact 4-manifold. All our analysis has been local and given a metric in a local coordinate chart as in (12), compactness is always an open question. The property of compactness depends on the range of coordinates that we may assign to the local coordinates. We have not done that, but the fact that the metric coefficients and curvature scalars are homogeneous functions of degree zero in the potential v and its derivatives, together with the presence of exponentials in the potential suggests that the metric could well be made compact by choosing a suitable domain of coordinates and parameters.

Assuming compactness, by virtue of its anti-self-dual curvature property, our solution saturates Hitchin's bound $|\tau| \leq (2/3)\chi$ [20] between the Euler characteristic χ and the Hirzebruch signature τ. By Hitchin's theorem [20] we know that only $K3$, and surfaces whose universal covering is $K3$ have this property.

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